Second Quarter Project

Pre-AP Calculus

You don't really recognize these relations when you are working with just the theory; in order to appreciate the intricacy, you need to actually handle and apply them, as we did here.

~excerpted from the concluding paragraphs of a student's math project (former student of Mr. O'B)

<u>Instructions</u>: Homework, quizzes, tests, and Supercorrections all provide opportunities for learning mathematics, but nothing beats a project for pulling it all together. This quarter, you will have the choice of one of two projects.

As before, you will be given some class time and some homework time to do your investigation and final write-up. You will use your graphing calculator and Geogebra as you investigate. Your final version **must be submitted in Google Docs**.

Keep in mind that there may be no single correct answer to a question, and you will be evaluated on the basis of your reasoning, justification, and communication skills. **Effective communication of ideas** is a very important component of mathematics.

The **rubric** for evaluation is on the next page. Your work will count as one of the two test grades for this unit.

<u>A note on collaboration</u>: You **may** discuss the content of this project with Mr. O'Brien, other students or anyone else but be sure to acknowledge any assistance received. Your final write-up must be your own.

Rough Draft Due: see iCal

Final Draft Due: see iCal

Quarter Project Rubric

Category	Poor	Fair	Good	Excellent
Presentation (10%) • Is the project visually pleasing? • Is there correct use of grammar and spelling?				
Mathematical precision and completeness (50%) • Are the solutions complete? • Are the solutions correct? • Has there been a correct use of mathematical notation?				
Verbal explanations (30%) • Are answers given in paragraph form without numbering? • Is there an introduction and conclusion? • Are the explanations complete and precise?				
Graphs (10%) • Are the graphs correct? • Are the graphs neat? • Is the graph labeled properly?				

General Comments:

Final Grade:________

The Population Problem

Population growth is a concern of many people around the world because of its impact on the environment, the food supply, and other limited resources. While this growth is very difficult to predict, particularly long-term growth, it is nonetheless necessary to make predictions in order to plan for the future. Mathematical models are often used to make reasonable predictions. These models can be quite complicated since population growth patterns are dependent upon many variable factors such as birth rate, death rate, and the age distribution of the population. However, relatively accurate information can be generated from simple mathematical models, particularly when their use is restricted to a short time period. The goal of this project is to help you understand simple mathematical models for population growth and what a growth rate number (something you might see in a newspaper article) represents.

In this project, we will simplify the situation by assuming that the world growth rate (birth rate − death rate) is constant within a calendar year.

1. A table produced by the U.S. Census Bureau gives the world population in 1996 as 5,771,938,438. It also states that the growth rate that year was 1.38% and that the increase in population in 1996 was 80,272,274.[21] We are interested in the relationship between the growth rate of 1.38% and the increase in population of 80,272,274. One possibility (perhaps the one that occurs to most people) is that the annual growth rate means that the world's population will increase by 1.38% of the existing population. What is 1.38% of 5,771,938,438? What is the difference between the answer you obtained and the increase given by the Census Bureau?

2. You should have found that your calculation from question 1 produced a significant underestimate. By calculating increase in this way, you are assuming that all of the increase takes place at the same time. This ignores the fact that as the population grows, there are more women of child-bearing age at the end of a given year than at the beginning, hence there are more children born at the end of that year.

(a) Let's try again, assuming that the population increases twice during the year. The semi-annual growth rate is $\frac{0.0138}{2}$ or 0.69%. We'll begin by finding a formula for the yearly increase in population.

[21] U.S. Bureau of Census, "Total Midyear Population for the World: 1950-2050," *International Data Base*, http://www.census.gov/ftp/pub/ipc/www/worldpop.html, (2 June 1997).

i. Write a formula for the total population (original population + the population increase) after only one half year of growth. Give your answer in factored form.

ii. Building on your formula from part i, write a formula for the total population after a full year of growth. Again, give your answer in factored form.

iii. Subtract the original population to get a formula for the population increase.

(b) Using the formula from part(a)iii, what will be the yearly increase in population if we assume it increases twice a year? What is the difference between the answer you obtained and the increase given by the Census Bureau?

3. We're getting closer, but we still haven't discovered how to get the increase given by the Census Bureau. Extending the idea developed in question 2, let's look at a model where population increases monthly.

(a) What is the monthly growth rate?

(b) Write a formula (similar to that in question 2(a)iii) for the yearly increase in population assuming population increases monthly.

(c) Using your formula from part (b), compute the yearly increase in population if we assume population increases monthly. Again, compare it to the increase given by the Census Bureau. What is the difference?

4. By compounding monthly, we obtain an answer closer to the increase given by the Census Bureau, but it is still not quite right. This is because our population actually changes almost continuously which, believe it or not, will make this problem a bit easier.

(a) Consider the function $f(n) = \left(1 + \frac{1}{n}\right)^n$ where n is an integer. Explore what happens when n gets large. Do this by completing Table 1, rounding your answers to four decimal places.

n	$\left(1 + \frac{1}{n}\right)^n$
1	2
10	
100	
1000	
10,000	
100,000	
1,000,000	

Table 1

Notice that as n gets larger, $\left(1+\frac{1}{n}\right)^n$ increases. The value of $f(n)$ does not increase without bound but instead gets closer to a fixed number called e, an irrational number *approximately* equal to 2.7182818. Mathematically speaking, we say that the limit, as n goes to infinity, of $\left(1+\frac{1}{n}\right)^n$ is e. This is sometimes written as $\lim\limits_{n\to\infty}\left(1+\dfrac{1}{n}\right)^n = e$. We are interested in this limit since it can be used in our population problem. To answer questions 2(b) and 3(c), you should have used a formula similar to

$$P_i = P_0\left(1+\frac{r}{n}\right)^{nt} - P_0,$$

where P_i is the population increase, P_0 is the initial population, r is the annual growth rate, t is the time (one year in this case), and n is the number of times that the growth is compounded per year.

(b) Let's figure out what happens to this formula as n, the number of times the population increases each year, goes towards infinity. We want to show that $\lim\limits_{n\to\infty} P_0\left(1+\dfrac{r}{n}\right)^{nt} = P_0e^{rt}$. Let $k = \frac{n}{r}$ and give reasons for both of the steps in the argument below.

$$\begin{aligned} P_0\left(1+\frac{r}{n}\right)^{nt} &= P_0\left(1+\frac{1}{k}\right)^{krt} \\ &= P_0\left[\left(1+\frac{1}{k}\right)^{k}\right]^{rt} \end{aligned}$$

As n increases to infinity, k does also since $k = \frac{n}{r}$ and r is fixed. Therefore, $\lim\limits_{n\to\infty} P_0\left(1+\dfrac{r}{n}\right)^{nt}$ $= \lim\limits_{k\to\infty} P_0\left[\left(1+\dfrac{1}{k}\right)^{k}\right]^{rt} = P_0e^{rt}$. So when we assume that our population grows continuously, the formula becomes $P_i = P_0e^{rt} - P_0$.

(c) Use the formula for continuous population growth to compute the population increase and compare it to that given in the census data. Compute the difference.

(d) Note that your answer still does not exactly match the census data even though it is significantly better than when we first started. This is because the growth rate number has been rounded to the nearest hundredth of a percent. In some cases, even a little rounding can make a big difference. Since 1.38% is rounded to the nearest hundredth of a percent, the yearly growth rate given could actually be anywhere from 1.375% to 1.385%. Using 1.375%, and the final formula from part (b), determine the effect of rounding on the accuracy of our approximation of the population increase.[22] [Hint: To do this, subtract the value you get for P_i

[22] We chose the smaller number since our result was an underestimate.

using 1.375% from the value you got using 1.38%.] Is your answer from part (c) within this range?

5. Now that we've looked at a simple mathematical model used to find short-term increases in population, let's try to gain a better understanding of the effects of growth rate on population. To do this, we will look at another way to describe population growth, the time it takes the population to double.

(a) If the world population grows continuously at an annual rate of 1.38%, how long will it take to double in size?

(b) Write a function where growth rate is the input and the time needed to double in size (known as the doubling time) is the output.

6. Instead of looking at world population, we will now compare population growth rates from three different countries. In doing so, we will focus not on the increase in population, but on the size of the total population after a given period of time. A document produced by the United Nations Population Fund gives an annual growth rate of approximately 1% for Ireland, a growth rate of approximately 2% for Mexico, and a growth rate of approximately 3% for Ethiopia from 1990 to 1995. (For this period, the United States had a growth rate of 0.71%.)[23] For comparison purposes, assume that each of these countries had an initial population of 1,000,000.

(a) Assuming the growth rates of Ireland, Mexico, and Ethiopia stay constant at the rates mentioned above, complete Table 2.[24]

Country	Growth Rate	Population after 10 years	Population after 100 years	Time to double
Ireland	1%			
Mexico	2%			
Ethiopia	3%			

Table 2

(b) Graph the population for these three countries, on the same set of axes, with time on the horizontal axis and population on the vertical axis. Use a domain of 0 to 10 years.

[23]United Nations Population Fund, *Population and the Environment: The Challenges Ahead*, Banson Productions, 1991, pp 39-43.

[24]In reality, population growth rates do not stay constant. We will assume they do in order to simplify this problem.

(c) Graph the population for these three countries with a domain of 0 to 100 years.

(d) Describe the differences and similarities between the graphs in the 10-year window and those in the 100-year window. How are the shapes of the graphs in these two windows related to the resulting differences in populations of the three countries after 10 years and after 100 years?

(e) The populations of each of these countries in 1990 was not 1,000,000, but was actually much larger. In 1990, Mexico had a population of about 90,000,000 and Ethiopia had a population of about 50,000,000. If the growth rates of these two countries stay constant, determine the year that Ethiopia's population would exceed that of Mexico. Do this both graphically (include the graph as part of your answer) and symbolically.

7. If the growth rate of one country is double that of another country, describe what "double" means. Does it mean the population of the second country will always be double that of the first? Does it mean that the increase in population of the second will always be double that of the first? Exactly what is doubled? Use examples from previous questions or make up other examples to support your conclusions.

Buy Now, Pay Later

The power that time has on money can be quite surprising when interest is involved. Interest is money that is paid for the use of money. This can work in your favor when you are saving money and it is earning interest that is compounded. It can also work against you when you are borrowing money for a car, a house, or purchases bought using a credit card.

Payments on many loans, like a mortgage on a house, are done in equal monthly payments. The formula used to compute such payments is

$$M = \frac{iP}{1-(1+i)^{-n}} \tag{1}$$

where M is the periodic payment (usually monthly), P is the principal,[46] i is the annual interest rate divided by the number of times compounded per year (usually monthly), and n is the total number of payments you will make (the number of payments per year multiplied by the number of years).

To help us understand what's going on in Equation (1), let's start by looking at a simple case. Assume you borrowed \$1000 for one year at 12% compounded quarterly (quarterly means interest is added to your account four times during the year or every three months). What actually happens is that at the end of the first quarter you are charged 3% on the amount you owe, in this case \$30. Your payment must cover the interest payment plus some portion of the principal (the amount you borrowed in the first place). According to the formula given above, that payment is \$269.03. After you make that first payment you now owe \$1030 − \$269.03 = \$760.97. When the time comes to make your second payment, you are charged interest on the amount left, i.e. \$760.97. So your new interest charge for the second quarter is 0.03(\$760.97) = \$22.83. This happens two more times. Each time you are charged interest on the money you still owe until, on your final payment, you pay off all the money. Table 1 summarizes what happens.

Quarter	Interest charged	Amount owed	Payment	Amount owed after payment
1st	\$30	\$1030	\$269.03	\$760.97
2nd	\$22.83	\$783.8	\$269.03	\$514.77
3rd	\$15.44	\$530.21	\$269.03	\$261.18
4th	\$7.84	\$269.02	\$269.02	\$0

Table 1

[46] The initial amount of money you deposit or borrow is called the principal.

1. Use Equation (1) to determine the payments on \$3000 at 8% compounded quarterly for one year. Show how much interest and principal has been paid in each of the four payments. A schedule, such as Table 2, listing each of the four payments including the amount of interest and principal is called an *amortization* schedule.[47]

Quarter	Interest charged	Amount owed	Payment	Amount owed after payment
1st				
2nd				
3rd				
4th				

Table 2

2. Formulas such as the one given for finding payments don't just drop out of the sky nor are they only understandable by "math wizards." Before continuing to use this formula, let's derive it starting with a simple case. Let P be the amount borrowed, r be the annual interest rate, M be the monthly payment. Assume we'll make the payments quarterly and plan to pay back the loan by the end of the year. The interest rate, i, we will be using is actually $\frac{r}{4}$ since we are making the payments quarterly.

At the end of the first quarter, we owe $P + iP = (1+i)P$. We then make our payment, M, leaving us with a balance of $(1+i)P - M$. At the end of the second quarter, we will owe

$$((1+i)P - M) + i((1+i)P - M) = (1+i)[(1+i)P - M] = (1+i)^2P - (1+i)M. \quad (2)$$

(a) Continuing this process, **show the steps** to get

$$(1+i)^4P - (1+i)^3M - (1+i)^2M - (1+i)M - M \quad (3)$$

at the end of the fourth quarter.

(b) Since this process paid off the whole loan, we can set it equal to zero.

$$(1+i)^4P - (1+i)^3M - (1+i)^2M - (1+i)M - M = 0. \quad (4)$$

Show the steps needed to get from Equation (4) to

$$P = \left(\frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4}\right)M. \quad (5)$$

[47]The word *amortization* contains the root word *mort*, the same root used in *mortician* and *mortal*, referring to death. An amortization schedule tells you how you "kill off" your loan.

(c) Equation (5) would look nicer if we could simplify the part in the large parentheses. Let's call that whole piece X, i.e.

$$X = \frac{1}{1+i} + \frac{1}{(1+i)^2} + \frac{1}{(1+i)^3} + \frac{1}{(1+i)^4}. \tag{6}$$

At this point we will use a procedure that is often used when finding the sum of a geometric series. Multiply both sides of Equation (6) by $(1+i)$ and subtract Equation (6) from the resulting equation. Lots of pieces should disappear! Now solve this equation for X and show that you get

$$X = \frac{1}{i}\left(1 - \frac{1}{(1+i)^4}\right).$$

(d) Place this result into Equation (5) and solve for M. How does this compare to the equation given for M at the beginning of this project? Explain why there is a difference.

3. Explain why, in order for this formula to be valid, it is necessary that how often you make your payment and how often the interest is compounded must be the same.

4. We will now look at each of the three different variables involved in our payment formula.

(a) Suppose you are buying a house and borrow \$80,000 for 30 years. You will be making 360 monthly payments. Put these values in Equation (1) and graph the resulting function so that the input is your annual rate, r, and the output is your monthly payment. [Note: $i = \frac{r}{12}$.] What is the monthly payment if the annual interest rate is 6%? 9%? 12%?

(b) This time, let $i = \frac{9\%}{12}$ and keep the 360 monthly payments, but let the principal vary. Put these values in Equation (1) and graph the resulting function so that the input is the principal and the output is the monthly payment. What is the monthly payment if the principal is \$40,000? \$80,000? \$120,000?

(c) Finally, let $i = \frac{9\%}{12}$ and let $P = \$80,000$, but let the number of payments vary. Put these values in Equation (1) and graph the resulting function so that the input is the number of payments and the output is the monthly payment. What is the monthly payment if you have the loan for 10 years? 20 years? 30 years?

(d) Based on your work above, write a brief description of what happens to the payment as each of the three variables changes while the others are held fixed.

5. Suppose you are planning on buying a house and determine that you can afford about a \$600 monthly house payment. Assume the current annual interest rate is 9%. You have \$3,000 saved for the downpayment.

(a) Show that if you take out a 15 year loan, you should be planning to buy a home that costs about \$62,000. Show that if you take out a 30 year loan, you should be planning to buy a home that costs about \$77,500.

(b) Suppose you purchase the \$62,000 home with a 15 year mortgage. What is the total price you will have paid for this house at the end of that 15 years? How much of this was interest?

(c) Suppose you purchase the \$77,500 home with a 30 year mortgage. What is the total price you have paid for this house at the end of that 30 years? How much of this was interest?

(d) What are your advantages in taking out a 15 year loan? What are your advantages of taking out a 30 year loan?